DSP : Lec 7

quiz

① using Z.T : find $y(n)$ in closed form for the difference eqn:-

$$y(n) = x(n-2) - x(n) + 0.5\, y(n-1)$$

where $x(n) = \begin{cases} 0 & \text{at } n \text{ even} \\ 1 & \text{at } n \text{ odd} \end{cases}$

② $x(n) = \left\{-1, 1, 0, \frac{1}{2}, \frac{-1}{2}\right\}$

Find * $x((-n))_5$ * $x((n-2))_5$

* $x((2-n))_5$

Sol

②

$$x((-n))_5 = \left\{-1, \frac{-1}{2}, \frac{1}{2}, 0, 1\right\}$$

$$x((n-2))_5 = \left\{\frac{1}{2}, \frac{-1}{2}, -1, 1, 0\right\}$$

$$x((2-n))_5 = \left\{0, 1, -1, \frac{-1}{2}, \frac{1}{2}\right\}$$

$\boxed{1}$ $Y(z) = z^{-2} X(z) - X(z) + 0.5\, z^{-1} Y(z)$

$$Y(z) - 0.5\, z^{-1} Y(z) = z^{-2} X(z) - X(z)$$

$$Y(z) = \frac{z^{-2} - 1}{1 - 0.5\, z^{-1}}\, X(z)$$

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

<u>if $n \to$ odd</u>

$$X(z) = \sum_{n=0} z^{-n} = z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$= \frac{z^{-1}}{1 - z^{-2}}$$

$$Y(z) = \frac{z^{-2} - 1}{1 - 0.5\, z^{-1}} * \frac{z^{-1}}{1 - z^{-2}} = \frac{z^{-2} - 1}{1 - 0.5\, z^{-1}} * \frac{z^{-1}}{-(z^{-2} - 1)}$$

$$= \frac{-z^{-1}}{1 - 0.5\, z^{-1}} = \frac{-1 * z * z^{-1}}{z - 0.5}$$

$$y(n) = -(0.5)^{n-1}\, u(n-1)$$

* Circular Convolution

note

Linear Convolution

$$y(n) = x_1(n) * x_2(n) = \sum_{K=-\infty}^{\infty} x_1(K) \cdot x_2(n-K)$$

$$= \sum_{K=-\infty}^{\infty} x_1(n-K) \cdot x_2(K)$$

* If the sequences (Discrete time sequences) are periodic, then the Convolution is circular.

$$y(n) = x_1(n) \; Ⓝ \; x_2(n)$$

where $x_1(n)$, $x_2(n)$ are two periodic sequences every N samples.

Ⓝ ⇒ the symbol of circular Convolution.

$$y(n) = x_1(n) \; (N) \; x_2(n)$$

$$= \sum_{m=0}^{N-1} x_1(m) \cdot x_2(n-m)$$

$$= \sum_{m=0}^{N-1} x_1(n-m) \cdot x_2(m)$$

$$= \text{IDFT} \left\{ X_1(K) \cdot X_2(K) \right\}$$

$X_1(K) \rightarrow$ DFT for the sequence $x_1(n)$

$X_2(K) \rightarrow$ " " " " $x_2(n)$

$\boxed{EX}$ Compute the circular Convolution for:-

$x_1(n) = \{2, 1, 2, 1\}$ and $x_2(n) = \{1, 2, 3, 4\}$

for $N = 4$ → circular Convolution

or find $x_1(n) \; (N) \; x_2(n)$

Sol

$$y(n) = x_1(n) \; (4) \; x_2(n) = \sum_{m=0}^{3} x_1(m) \cdot x_2((n-m))_4$$

	m	0	1	2	3	
	$X_1(m)$	2	1	2	1	
	$X_2(m)$	1	2	3	4	فائدة التعويض بـ n إنى هعرف أجيب
$n=0$	$X_2((-m))_4$	1	4	3	2	$\Rightarrow y(0)$
$n=1$	$X_2((1-m))_4$	2	1	4	3	$\Rightarrow y(1)$
$n=2$	$X_2((2-m))_4$	3	2	1	4	$\Rightarrow y(2)$
$n=3$	$X_2((3-m))_4$	4	3	2	1	$\Rightarrow y(3)$

$$y(0) = 1*2 + 1*4 + 3*2 + 2*1 = 14$$

$$y(1) = X_2((1-m))_4 * X_1(m) = 2*2 + 1*1 + 2*4 + 1*3 = 16$$

$$y(2) = X_2((2-m))_4 * X_1(m) = 14$$

$$y(3) = X_2((3-m))_4 * X_1(m) = 16$$

$$y(n) = \{14, 16, 14, 16\}$$

ممكن طريقة ثانية

$$y(n) = x_1(n) \; ④ \; x_2(n)$$

$$x_2((-m)) = \begin{pmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}_{\substack{N*N \\ 4*4}} \underbrace{\begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}}_{x_1(n)}{}_{\substack{N*1 \\ 4*1}}$$

$$= \begin{pmatrix} 14 \\ 16 \\ 14 \\ 16 \end{pmatrix} \Rightarrow y(n) = \{14, 16, 14, 16\}$$

or $y(n) = \sum_{m=0}^{3} x_2(m) \cdot x_1(n-m)$

$$= \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 14 \\ 16 \\ 14 \\ 16 \end{pmatrix}$$

another solution

$$y(n) = x_1(n) \; \textcircled{4} \; x_2(n) = IDFT\left(X_1(K) \cdot X_2(K)\right)$$

$\boxed{1}$ Compute DFT for $x_1(n)$ & $x_2(n)$

$$X_1(K) = \begin{pmatrix} 6 \\ 0 \\ 2 \\ 0 \end{pmatrix} , \quad X_2(K) = \begin{pmatrix} 10 \\ -2+2J \\ -2 \\ -2-2J \end{pmatrix}$$

$$X_1(K) \cdot X_2(K) = \begin{pmatrix} 6 \\ 0 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 10 \\ -2+2J \\ -2 \\ -2-2J \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \\ -4 \\ 0 \end{pmatrix}$$

$$y(n) = IDFT\left(X_1(K) \cdot X_2(K)\right)$$

$$= \begin{pmatrix} 14 \\ 16 \\ 14 \\ 16 \end{pmatrix}$$

Prove the following Properties

$\boxed{1}\ W_N^{K+N} = W_N^K$ $\qquad$ $\boxed{2}\ W_N^{K+\frac{N}{2}} = -W_N^K$

$\boxed{3}\quad W_N^2 = W_{N/2}$

$$W_N = e^{-J\frac{2\pi}{N}} \quad \text{"twiddle factor"}$$

$\boxed{1}\ W_N^{K+N} = W_N^K \cdot W_N^N$

$$W_N^N = e^{-J\frac{2\pi}{N}*N} = e^{-J2\pi} = 1$$

$$\therefore W_N^{K+N} = W_N^K \quad \#$$

$\boxed{2}\ W_N^{K+\frac{N}{2}} = -W_N^K$

$$W_N^{K+\frac{N}{2}} = W_N^K * W_N^{N/2}$$

$$W_N^{N/2} = e^{-J\frac{2\pi}{N}*\frac{N}{2}} = e^{-J\pi} = \cos(180) - J\sin(180)$$
$$= -1$$

$$\therefore W_N^{K+\frac{N}{2}} = -W_N^K$$

$\boxed{3}$ $W_N^2 = W_{N/2}$

$$W_{N/2} = e^{-J\frac{2\pi}{N/2}} = \left(e^{-J\frac{2\pi}{N}}\right)^2$$

$$= W_N^2$$

* Fast Fourier Transform (FFT) Algorithms :-

↳ are many algorithms used to reduce the Complex Computation for DFT

$\boxed{1}$ Radix-2 DIT FFT

DIT ⇒ Decimation (spliting) in time.

⇒ for periodic sequence $x(n)$, the discrete Fourier transform is $X(K)$ which Periodic N samples.

$$\underbrace{x(n)}_{\substack{\downarrow \\ \text{Discrete} \\ \text{time Sequence}}} \xrightarrow[N]{DFT} \underbrace{X(K)}_{DFT}$$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$$

الصورة القديمة.

Where: $W_N = e^{-J\frac{2\pi}{N}}$

For Radix-2 DIT FFT algorithm:-

$$X(K) = \underbrace{\sum_{n=\text{even}}}_{\frac{N}{2}\text{ point DFT}} + \underbrace{\sum_{n=\text{odd}}}_{\frac{N}{2}\text{ point DFT}}$$

For ex

$N=6 \Rightarrow x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5)\}$

$x_1(n) = \{x(0), x(2), x(4)\} \rightarrow$ even numbered sequence

$x_2(n) = \{x(1), x(3), x(5)\} \rightarrow$ odd numbered seq.

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2Kn} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{K(2n+1)}$$

even numbered sequence

$$W_N^{2Kn} \equiv \left(W_N^{Kn}\right)^2$$

$$W_N^{K(2n+1)} = W_N^{2Kn} \cdot W_N^{K}$$

نصف ال (sequence)

لكن الجزء ال (even)

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_{N/2}^{Kn} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_{N/2}^{Kn} \cdot W_N^{K}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_{N/2}^{Kn} + W_N^{K} \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_{N/2}^{Kn}$$

assume: $f_1(n) = x(2n)$

$f_2(n) = x(2n+1)$

معناها اني هضع $x(2n)$ ف (sequence) اسه $f_1(n)$.

$$X(K) = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} f_1(n) W_{N/2}^{Kn}}_{F_1(K)} + W_N^{K} \sum x(2n+1) W_{N/2}^{Kn}$$

$$\boxed{X(K) = F_1(K) + W_N^K F_2(K)} \longrightarrow \boxed{1}$$

where:

$F_1(K)$ is DFT for $f_1(n) = x(2n)$

$F_2(K)$ is DFT for $f_2(n) = x(2n+1)$

$F_1(K) \longrightarrow \frac{N}{2}$ Point DFT

$F_2(K) \longrightarrow \frac{N}{2}$ Point DFT

For eq(1) $\Rightarrow K = 0, 1, \ldots, \frac{N}{2} - 1$

$\hookrightarrow X(K)$ has $\frac{N}{2}$ Point DFT

$$X\left(K + \frac{N}{2}\right) = F_1\left(K + \frac{N}{2}\right) + W_N^{K + \frac{N}{2}} F_2\left(K + \frac{N}{2}\right)$$

$$\boxed{X\left(K + \frac{N}{2}\right) = F_1(K) - W_N^K F_2(K)}$$

$$K = 0, 1, \ldots, \frac{N}{2} - 1$$

* For large values of N, this operation is repeated until we reach to 2-Point DFT Computation.

Ex: $x(n) = \{x(0), x(1)\} \Rightarrow N = 2$

$$X(K) = \sum_{n=0}^{1} x(n)\, W_2^{Kn}$$

$$\cancel{= x(0)\, \underbrace{W_2^0}_{1} + x(1)\, \underbrace{W_2^1}_{-1}}$$

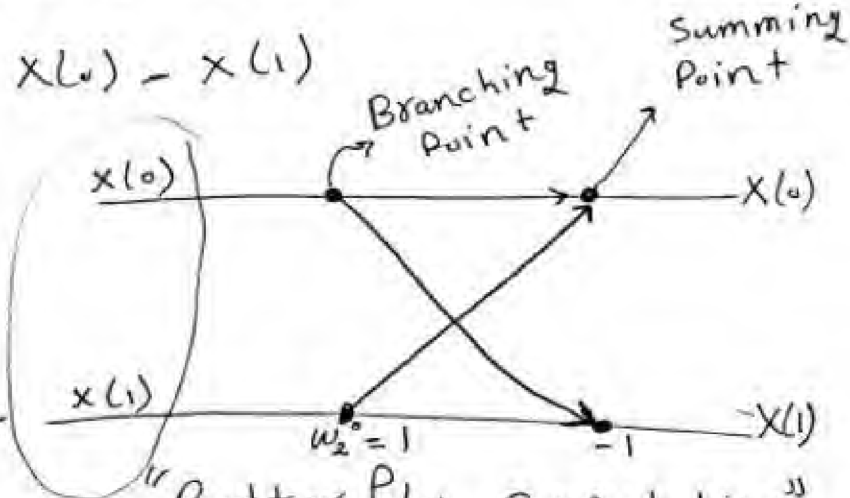
$$W_2^1 = e^{-J\frac{2\pi}{2}} = e^{-J\pi} = -1$$

$$X(K = x(0)\, W_2^0 + x(1)\, W_2^K$$

$$\cancel{X(K) = x(0) - x(1)}$$

$$K = 0 \Rightarrow X(0) = x(0) + x(1)$$

$$K = 1 \Rightarrow X(1) = x(0) - x(1)$$



الشكل ده بيحقق المعادلتين

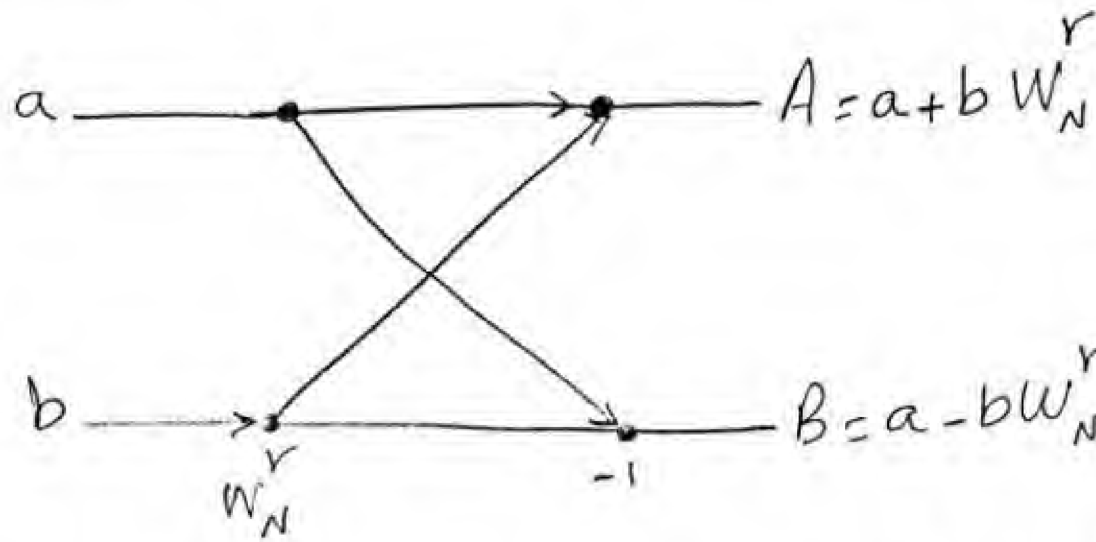
$X(1)$ ، $X(0)$

2-Point Discrete time sequence

"Butter fly Computation"

$X(0), X(1) \rightarrow$ 2-Point DFT.

In general



"general form for Butterfly computation"

$\boxed{Ex}$ For $N = 4$

$$X(K) = F_1(K) + W_N^K F_2(K) \rightarrow ①$$

$$X\left(K + \frac{N}{2}\right) = F_1(K) - W_N^K F_2(K) \rightarrow ②$$

$$K = 0, 1, \ldots, \frac{N}{2} - 1$$

where:

$f_1(n) = x(2n) \rightarrow$ even numbered sequence

$f_2(n) = x(2n+1) \rightarrow$ odd " "

$$X(K) = F_1(K) + W_4^K \, F_2(K) \longrightarrow (1)$$

$$X(K+2) = F_1(K) - W_4^K \, F_2(K) \longrightarrow (2)$$

$K = 0, 1$

eqn (1)

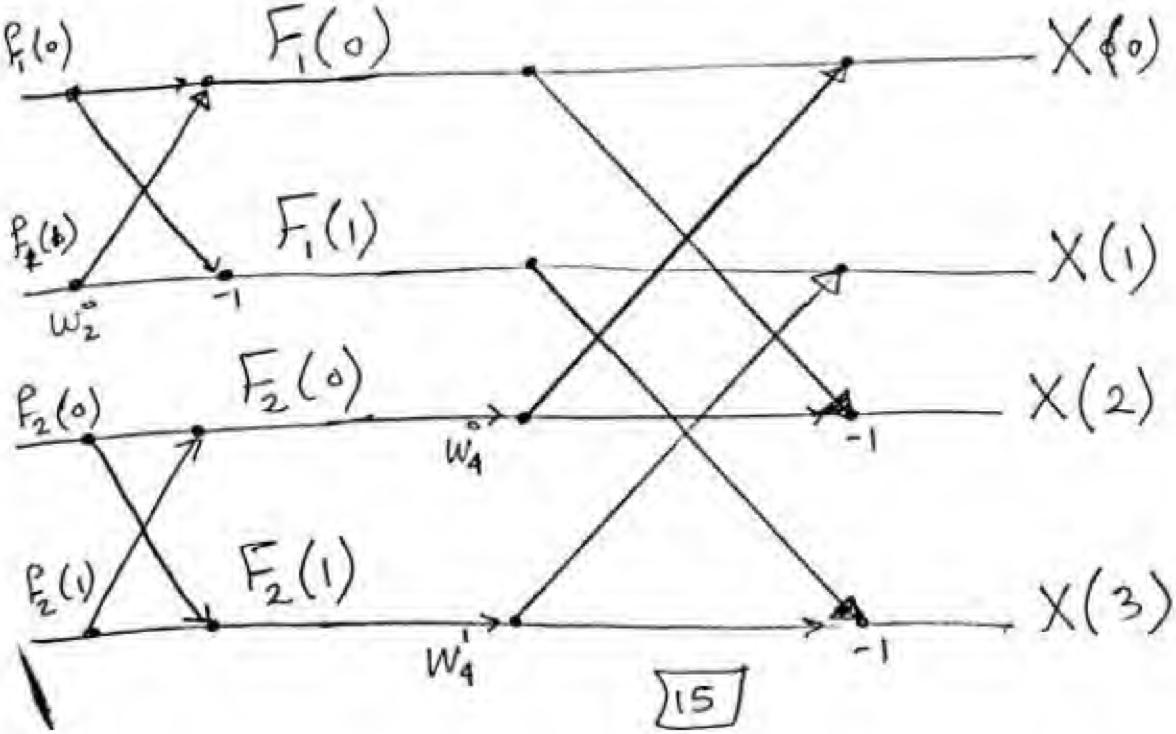
$K = 0 \Rightarrow X(0) = F_1(0) + W_4^0 \, F_2(0)$ → 2-Point DFT

$K = 1 \Rightarrow X(1) = F_1(1) + W_4^1 \, F_2(1)$

in eqn 2

$K = 0 \Rightarrow X(2) = F_1(0) - W_4^0 \, F_2(0)$ → 2-Point DFT

$K = 1 \Rightarrow X(3) = F_1(1) - W_4^1 \, F_2(1)$



ـــ في الرسم السابق كل علامة x تمثل (Butter fly)

ـــ $f_1(n) = x(2n) \to n=0,1$

$f_1(n) = \{x(0),\ x(2)\}$

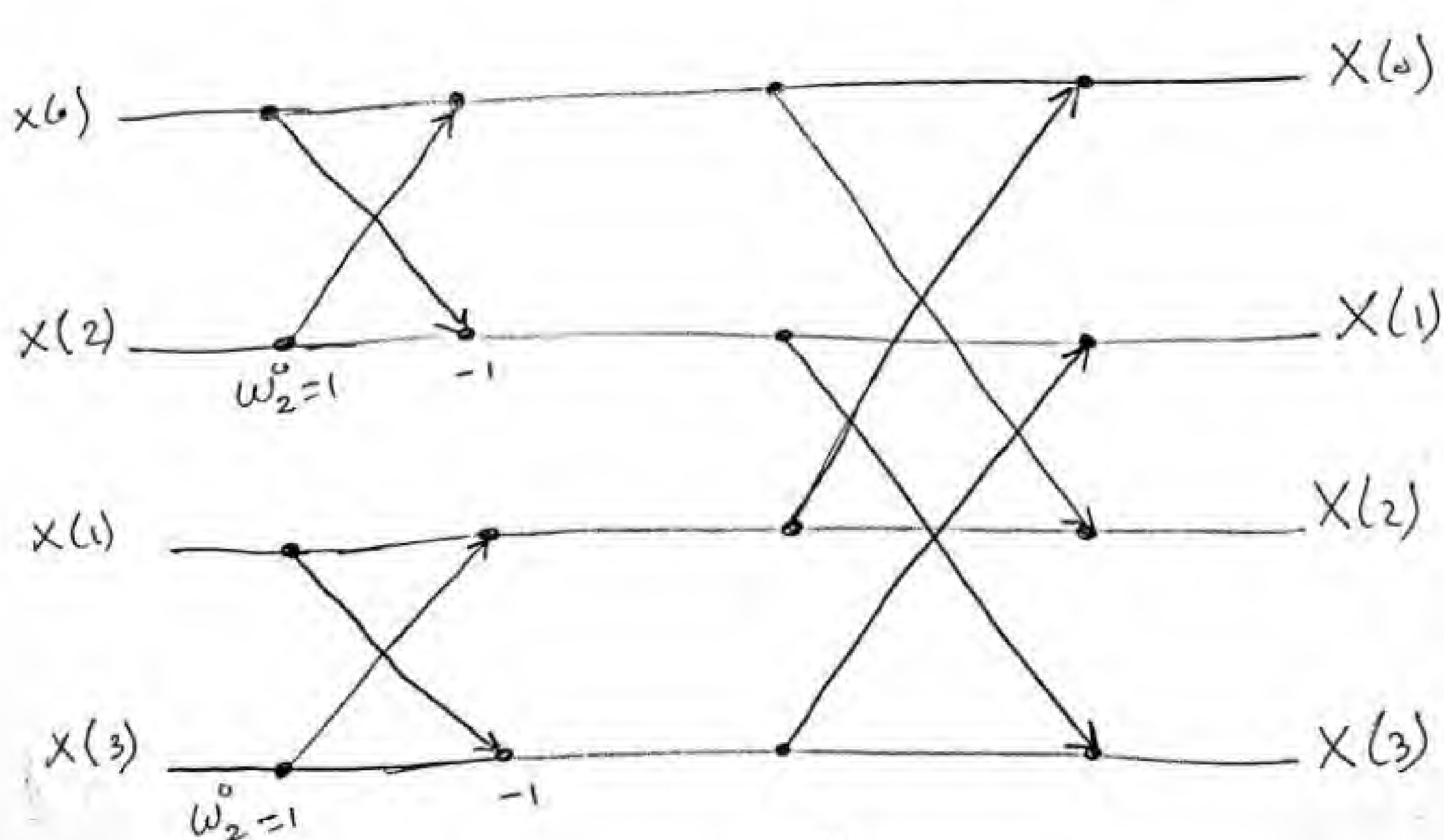
$f_2(n) = x(2n+1) \quad , n=0,1$

$= \{x(1)\ , x(3)\}$

$f_1(0) = x(0), \quad f_1(1) = x(2)$

$f_2(0) = x(1) \quad f_2(1) = x(3)$

ـــ ممكن ترسمها كده



$\boxed{Ex}$ Find radix-2 DIT FFT for

$x(n) = \{0, 1, 2, 3\}$